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The Coulomb branch of gauge theory from rotating branes

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Abstract: At zero temperature the Coulomb Branch of $\mathcal{N} = 4$ super Yang-Mills theory is described in supergravity by multi-center solutions with D3-brane charge. At finite temperature and chemical potential the vacuum degeneracy is lifted, and minima of the free energy are shown to have a supergravity description as rotating black D3-branes. In the extreme limit these solutions single out preferred points on the moduli space that can be interpreted as simple distributions of branes — for instance, a uniformly charged planar disc. We exploit this geometrical representation to study the thermodynamics of rotating black D3-branes. The low energy excitations of the system appear to be governed by an effective string theory which is related to the singularity in spacetime.

Keywords: Black Holes in String Theory, D-branes, Black Holes.

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Contents

1. Introduction

It has become evident that many questions concerning large N gauge theories can be answered via supergravity [1]. Among the many applications of this approach, one can compute correlation functions of the gauge theory [2, 3], map out its phase structure and study general thermodynamic properties [4, 5, 6, 7, 8]. In almost all investigations conducted so far the gauge theories have been studied at the origin of moduli space, where there is an enhanced superconformal symmetry. At this special point the group theory of the superconformal algebra can be brought to bear, leading to many exact results even at strong coupling. Such field theories are described by gravity in anti-de Sitter space (AdS).

However, it is also desirable to extend the investigations away from the origin of moduli space. Giving expectation values to certain scalar fields moves one onto the Coulomb branch — a space of maximally supersymmetric but nonconformal vacua. We would like to know how the physics of the Coulomb branch can be addressed using supergravity. As a complementary motivation, for the purposes of understanding quantum gravity via gauge theory, one would like to have access to geometries besides just AdS. While linearized perturbations around AdS are well understood in both the gravity and gauge theory contexts, less has been said about large departures from AdS which involve the nonlinear aspects of gravity in an important way.

In this work we study the Coulomb branch of $\mathcal{N}=4$ super Yang-Mills theory. The origin of moduli space represents a collection of coincident D3-branes, whose near horizon geometry is $AdS_5 \times S^5$. Separating some of the branes in the transverse space moves the gauge theory onto the Coulomb branch. One would like to associate each point on the Coulomb branch with some new supergravity solution which is *asymptotically* $AdS_5 \times S^5$. Indeed, solutions representing multiple D3-branes at distinct points are well known, and referred to as multi-center metrics. Such supergravity solutions are written in terms of a harmonic function which has sources at the positions of the D3-branes. Thus, as we review in section 2, there is a simple one-to-one correspondence between points on the Coulomb branch and multi-center geometries (see also [1, 9, 10, 11, 12]).

Once the moduli space has been understood we consider gauge theory on the Coulomb branch at finite temperature. Then supersymmetry is broken and the vacuum degeneracy is removed; the gauge theory settles to points which minimize the free energy of the system. If temperature is the only thermodynamic parameter present, then one expects that the free energy will be minimized at the origin of moduli space: away from the origin, certain fields acquire masses and so contribute less to the entropy. To investigate the theory away from the origin we can include a chemical potential for charge under the global SO(6) R symmetry group of the $\mathcal{N}=4$ theory. Physically, this means adding angular momentum in the space transverse to the collection of D3-branes. Now we expect the free energy to be minimized away from the origin, and we would like to know precisely where. At weak coupling this can be answered through a perturbative computation of the free energy. At strong coupling we turn to supergravity for the answer.

At zero temperature we noted that there was a supergravity solution corresponding to each point on the Coulomb branch. The situation at finite temperature is entirely different. From the no-hair theorems we expect that there is a unique solution corresponding to each value of the mass, charge, and angular momentum. The solution of interest to us – nonextremal, rotating D3-branes – has been derived up to duality in [13], and it is given in an appendix A. According to our discussion, this solution represents excited states of the gauge theory at points in moduli space that minimize the free energy.

We identify these special points in moduli space in section 3 by taking the extreme limit of the rotating D3-brane geometry [14], since in this limit the solution must reduce to some multi-center solution. The answer turns out to be very simple. It depends on how many of the three independent SO(6) angular momentum parameters $l_{1,2,3}$ are nonvanishing. $(l_1 \neq 0, l_{2,3} = 0)$ corresponds to a uniform distribution of D3-branes on a two dimensional disk of radius l_1; $(l_{1,2} \neq 0, l_3 = 0)$ corresponds to a distribution on a three dimensional ellipsoidal shell, $(y_1^2 + y_2^2)/l_1^2 + (y_3^2 + y_4^2)/l_2^2 = 1$. The general case $l_{1,2,3} \neq 0$ is more subtle, as we discuss; at this point we simply note that the distribution is contained within a five dimensional ellipsoid. For completeness, we also derive the extreme limits of rotating M2 and M5 brane solutions; the results are qualitatively similar.

From the nonextreme solutions one can deduce various thermodynamic quantities, and these become predictions for the large N gauge theory on the Coulomb branch. We discuss these predictions in section 4. In some respects the features revealed by black hole thermodynamics agree qualitatively with expectations for the gauge theory; in other respects, they reveal some intriguing new aspects. For example, in the case where only one angular momentum parameter l_1 is non-zero, and the extremal configuration consists of branes distributed on a disk, we find that low-energy excitations above the extremal state are governed by an effective string theory with a string tension that is determined by the Higgs VEV.

The extreme solutions we discuss are typically singular on the surface containing the D3-branes. For a disk configuration we show that such singularities have a gauge theory interpretation: as the scale decreases along the renormalization group flow, the effective field theory changes from the ultraviolet $\mathcal{N} = 4$ theory to the infra-red effective string theory mentioned above. One expects the transition between the two to occur at a scale of order the string tension in the effective string theory. Using a Wilson loop probe we argue, in section 5, that this energy scale in the gauge theory corresponds to the radial position of the singularity in the bulk theory.

A particularly interesting multi-center solution is obtained by distributing the D3-branes uniformly on a five sphere of radius l. Then the supergravity solution is given by flat ten dimensional spacetime inside the sphere, and $AdS_5 \times S^5$ outside. In the gauge theory, we expect that the inner region is represented by the low energy effective theory valid below scale l.

This paper is organized as follows. In the following section 2 we analyze the Coulomb branch and its relation to multi-center solutions; we also consider the connection with the AdS/CFT correspondence using linear perturbation theory. Next, in section 3, we discuss the extreme rotating solutions to supergravity, and their underlying brane distribution. Section 4 discusses the thermodynamics of rotating near-extreme D3-brane black holes. Following that, in section 5 we discuss the

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connection between singularities in spacetime and renormalisation group flows in the boundary theory. We conclude the paper by mentioning some connections with other recent work. The nonextreme rotating $D3$-brane solution is given in the appendix A.

2. The Coulomb branch and multi-center solutions

In this section we will establish a one-to-one correspondence between points on the Coulomb branch of $\mathcal{N}=4$ super Yang-Mills theory and multi-center D3-brane solutions of supergravity. We begin by reviewing some standard facts about the moduli space of the gauge theory, in particular its parametrization by eigenvalues of Higgs fields or by gauge invariant operators. An analogous discussion of multi-center supergravity solutions follows. Next, we consider the relation between these ideas and the AdS/CFT correspondence relating the asymptotic behavior of bulk fields to gauge theory operators; this discussion will exhibit an important subtlety. Finally, we discuss a supergravity solution which includes a region of flat spacetime contained within a spherical shell of D3-branes.

2.1 The Coulomb branch

$\mathcal{N}=4$ SU(N) super Yang-Mills theory includes the scalar fields Φ_i ($i=1,\ldots,6$) which transform in the vector representation of the SO(6) R symmetry group, and the adjoint representation of SU(N). The Coulomb branch corresponds to giving these fields expectation values subject to the flatness conditions $[\Phi_i,\Phi_j]=0$. Upon diagonalizing the fields, the moduli space is parametrized by the $6N$ eigenvalues $y_i^{(a)}$ ($a=1,\ldots,N$):

$$\Phi_i = \begin{pmatrix} y_i^{(1)} & & \\ & \ddots & \\ & & y_i^{(N)} \end{pmatrix} . \tag{2.1}$$

Tracelessness of Φ_i reduces the number of independent eignenvalues to $6(N-1)$. At generic points the gauge symmetry is broken to $\mathrm{U}(1)^{N-1}$. The low energy effective theory valid below the scale set by the VEVs is obtained by integrating out massive off-diagonal degrees of freedom.

For later comparison with the supergravity description we note that when N is large, instead of giving a list of $6N$ eigenvalues, it can be more convenient to give an approximate description in terms of a continuous distribution. Thus we let $\sigma(\vec{y})$ denote the density of eigenvalues in the six dimensional $\vec{y}$ plane.

An alternative way to parametrize the moduli space is in terms of gauge invariant operators. A complete set is given by the symmetric, traceless polynomials:

$$\mathcal{O}_{(i_1\cdots i_p)} = \mathrm{Tr}\ \Phi_{(i_1}\cdots\Phi_{i_p)}\,, \qquad p=2,\ldots,N\,. \tag{2.2}$$

Operators $\mathcal{O}_{(i_1 \cdots i_p)}$ for $p > N$ can be expressed in terms of operators with $p \leq N$. The descriptions in terms of eigenvalues and in terms of gauge invariant operators contain the same information; given the eigenvalues one can work out the values of the polynomials, and vice versa. For purposes of comparing with the supergravity solutions below we also note that the Coulomb branch preserves $\mathcal{N} = 4$ supersymmetry and the gauge coupling is not renormalised along this branch.

2.2 Multi-center D3-brane solutions

Configurations of the form:

$$ds^2 = H_{D3}^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2) + H_{D3}^{\frac{1}{2}} \sum_{i=1}^{6} dy_i^2 \,, \tag{2.3}$$

$$C^{(4)} = (H_{D3}^{-1} - 1)\; dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \,, \tag{2.4}$$

are solutions to type IIB supergravity for *any* harmonic function $H_{D3}(\vec{y})$. A general harmonic function has the integral representation:

$$H_{D3}(\vec{y}) = 1 + R^4 \int d^6y' \frac{\sigma_{D3}(\vec{y'})}{|\vec{y} - \vec{y'}|^4} \,, \tag{2.5}$$

where R^4 is the constant:

$$R^4 = 4\pi g_s \alpha'^2 N_{D3} \;. \tag{2.6}$$

The distribution function σ_{D3} is normalized as:

$$\int d^6y' \; \sigma_{D3}(\vec{y'}) = 1 \,, \tag{2.7}$$

so N_{D3} is the total number of D3-branes.

It is apparent that the space of multi-center solutions corresponds to the possible distribution functions one can write down. The distribution:

$$\sigma_{D3}(\vec{y}) = \frac{1}{N} \sum_{a=1}^{N} \delta^{(6)}(\vec{y} - \vec{y}^{(a)}) \,, \tag{2.8}$$

corresponds to D3-branes located at the discrete points $\vec{y}^{(a)}$. These points are naturally identified with the eigenvalues appearing in (2.1). Thus one expects that the gauge theory and supergravity configurations are dual descriptions.

In the classical supergravity limit the distribution need not be a sum of δ-functions, it may be continuous. Indeed, the solutions we obtain later from taking the extremal limit of certain nonextremal solutions will be of this sort. An important distinction is that while discrete distributions always give rise to nonsingular solutions of classical supergravity, the solutions obtained from continuous distributions may exhibit naked singularities, as we discuss in section 5. As mentioned previously, continuous distributions correspond in the gauge theory at finite N to specifying the approximate distribution of eigenvalues.

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Instead of directly giving the distribution $\sigma_{D3}(\vec{y})$ we can specify its moments:

$$\mathcal{O}_{(i_1 \cdots i_p)} = \int d^6 y \, y_{(i_1} \cdots y_{i_p)} \sigma_{D3}(\vec{y}) \, . \tag{2.9}$$

In terms of the moments, the harmonic function H_{D3} takes the form of an expansion in spherical harmonics. Spherical harmonics on S^5 can be expressed in terms of symmetric, traceless polynomials of $\vec{y}$, so we write:

$$H_{D3}(\vec{y}) = 1 + R^4 \sum_{p=0}^{\infty} \frac{\mathcal{O}_{(i_1 \cdots i_p)} Y_{(i_1 \cdots i_p)}(\Omega)}{|\vec{y}|^{4+p}} \, , \tag{2.10}$$

where Ω denotes angular coordinates on S^5. We note that in SU(N) gauge theory the first moment $\mathcal{O}_i$ actually vanishes, but we will not indicate that explicitly. The parametrization in terms of moments is analogous to the parametrization of the Coulomb branch by gauge invariant operators.

We see that the multi-center solutions, as described by the distribution σ_{D3} or its moments, are in one-to-one correspondence with points on the Coulomb branch of the gauge theory, as parametrized by eigenvalues or by gauge invariant operators. In addition, the solutions (2.3)–(2.4) preserve $\mathcal{N} = 4$ supersymmetry with a constant dilaton. These facts strongly suggest that the Coulomb branch of the gauge theory is equivalent to the space of multi-center solutions in the supergravity theory. In the following we will see that agreement in the linearized regime give further evidence for the correspondence.

2.3 Supergravity perturbations in $AdS_5 \times S^5$

The precise connection between gravity and gauge theory employs the near-horizon limit of the supergravity solutions; i.e. the 1 is omitted from the definition of the harmonic function H_{D3}, so that the resulting solutions are asymptotically $AdS_5 \times S^5$. The gauge theory operators are related to the deviations from $AdS_5 \times S^5$ as encoded in the asymptotic behavior of the supergravity fields [2, 3]. In the following we apply this formalism to the multi-center solutions in order to determine the expectation values of operators in the gauge theory. The goal is to verify the interpretation of multi-center solutions in the gauge theory as specifying points on the Coulomb branch of the moduli space, and in particular the precise map given in the discussion above.

The linearized supergravity perturbations on $AdS_5 \times S^5$ were classified in [15, 16] and their identification with gauge invariant operators of $\mathcal{N} = 4$ super Yang-Mills was made in [3, 17, 18]. We focus on the fields corresponding to perturbations of the ten dimensional metric, as they are the most relevant for our purposes. To make contact with [15] one writes the metric as:

$$g_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} + h_{\hat{\mu}\hat{\nu}} \, , \tag{2.11}$$

where $\hat{g}_{\hat{\mu}\hat{\nu}}$ is the $AdS_5 \times S^5$ metric and $h_{\hat{\mu}\hat{\nu}}$ is the perturbation. To classify the perturbations, we divide the hatted indices into μ, denoting coordinates in AdS_5, and α, the coordinates on S^5. Thus the various perturbations are written as $h \equiv \hat{g}^{\alpha\beta} h_{\alpha\beta}$, $h_{(\alpha\beta)}$, $h_{\mu\alpha}$, $h'_{\mu\nu} \equiv h_{\mu\nu} + \frac{1}{3}\hat{g}_{\mu\nu} h$, where $(\alpha\beta)$ denotes a symmetric, traceless combination of indices. The next step is to expand the perturbations in S^5 harmonics and to insert the expansions into the quadratic order supergravity action, to obtain a set of decoupled linear wave equations in AdS_5. Each independent partial wave component corresponds to a specific gauge theory operator. Thus by representing the multi-center solutions as perturbations around the background metric, it seems possible to identify which operators in the gauge theory have nonzero values. However, while this procedure works straightforwardly for the leading nontrivial harmonics, for the higher harmonics one needs to confront the fact that the multi-center solutions are actually solutions of *nonlinear* supergravity.

Let us make this more concrete. From the form of a multi-center solution, it is clear that the only perturbations which are potentially nonvanishing are h and $h'_{\mu\nu}$. Let us first focus on h. Expanding h in harmonics on S^5 (where we now use a single k as shorthand for the indices $i_1, \ldots, i_k$):

$$h = \sum_k h_k(x^\mu) Y_k(\Omega) \,, \tag{2.12}$$

one finds that to linear order the k'th harmonic obeys [15]:

$$[\Box_{AdS_5} - k(k-4)] h_k = 0 \,. \tag{2.13}$$

The general, normalizable solution that is translationally invariant in the brane direction $x_0 \ldots x_3$ is:

$$h = \sum_k \frac{\tilde{Q}_k}{|\vec{y}|^k} Y_k(\Omega) \,. \tag{2.14}$$

As discussed in [3], the k'th harmonics of h are dual in the gauge theory to the k'th order symmetric, traceless polynomials $O_{(i_1 \cdots i_k)}$. In other words, as in [19], the presence of the perturbation (2.14) means that these gauge theory operators have expectation values:

$$< \mathcal{O}_{(i_1 \cdots i_k)} > \sim \; \tilde{Q}_k \,. \tag{2.15}$$

Thus one is studying the gauge theory at a point on the Coulomb branch. To add sources in the action for the operator $O_{(i_1 \cdots i_k)}$ one should in addition turn on a non-normalizable h perturbation.

We now try to apply this approach to the multi-center solutions; we'll see that the analysis is more subtle than one might have guessed. Consider a solution of the form (2.3) where the harmonic function is:

$$H_{D3} = \sum_{k=0}^{\infty} \frac{Q_k}{|\vec{y}|^{4+k}} Y_k(\Omega) \,. \tag{2.16}$$

Keeping only the Q_0 term gives $AdS_5 \times S^5$ for the metric; the perturbation h is obtained by subtracting this contribution:

$$h = \hat{g}^{\alpha\beta} h_{\alpha\beta} = \frac{|\vec{y}|^2}{Q_0^{1/2}} \left\{ \left(\sum_{k=0}^{\infty} \frac{Q_k}{|\vec{y}|^{4+k}} Y_k(\Omega) \right)^{1/2} - \left(\frac{Q_0}{|\vec{y}|^4} Y_0(\Omega) \right)^{1/2} \right\} \tag{2.17}$$

$$= \frac{1}{2|\vec{y}|} \frac{Q_1}{Q_0} Y_1(\Omega) + \frac{1}{2|\vec{y}|^2} \left(\frac{Q_2}{Q_0} Y_2(\Omega) - \frac{1}{4} \frac{Q_1^2}{Q_0^2} \left(Y_1(\Omega)\right)^2 \right) + \cdots \tag{2.18}$$

There is a mismatch between this expression and the form given in (2.14): the angular dependence of the terms are not spherical harmonics beyond the first term. Thus, beyond the lowest order the harmonics of h do not obey the homogeneous equation (2.13), but rather some inhomogeneous equation:

$$[\Box_{AdS_5} - k(k-4)] h_k \approx (h)^2 + (h)^3 + \cdots \tag{2.19}$$

where combinations of $k' < k$ harmonics appear on the right hand side (as do higher orders in the perturbations of other fields, including the five-form). The reason for this complication is clear: in the linearized approximation of [15] one — by definition — omits the source terms. However, one must verify that this is consistent, and this requirement translates into conditions on the underlying distribution function σ_{D3}. For a generic source of spatial extent $\sim l$ we expect moments of the order $Q_k/Q_0 \sim l^k$ and then the linear order perturbation theory does *not* suffice. However, the linear approximation is valid for specially prepared distributions, and it always applies to the lowest harmonic contributing to the expansion of h in (2.18).

In fact, in SU(N) gauge theory the linear harmonic actually vanishes, $Q_1 = 0$. Thus the leading nontrivial order is Y_2. In this case the next correction Y_3 cannot receive any corrections, because the terms of lower order are of even degree. However, the Y_4 can receive non-linear corrections, of order Y_2^2, and the discussion proceeds as before.

In cases for which the linearized approximation is valid, one can readily apply the methods of [20] to prove that the multi-center solutions are described in gauge theory by points on the Coulomb branch. The idea is to consider a non-normalizable mode perturbation of h_k around a given solution. On the supergravity side, upon integration by parts the change in action can be written as a boundary term proportional to the normal derivative of h_k. The variation of the gauge theory partition function is proportional to the expectation value of the operator $\mathcal{O}_{(i_1 \cdots i_k)}$. Equating these, one proves (2.15).

So far we have only considered the field h. However, similar considerations apply to other fields, in particular the perturbation $h'_{\mu\nu}$, which is dual to operators in the gauge theory involving products of Higgs fields with the energy-momentum tensor. From the above analysis, one can reliably study the lowest harmonic of $h'_{\mu\nu}$ and thus read off the value of the energy-momentum tensor. In fact, one should actually compute the field $\phi_{(\mu\nu)}$, defined in (2.44) of [15], which differs from $h'_{\mu\nu}$ by terms related to h and the five-form field. When these effects are taken into account, direct evaluation in the case of a multi-center solution reveals that $\phi_{(\mu\nu)}$ vanishes, indicating zero energy-momentum density in the gauge theory. This is consistent with the interpretation of being on the Coulomb branch which — being a supersymmetric configuration — has zero energy.

Thus the analysis of the multi-centered solutions in terms of perturbations of $AdS_5 \times S^5$ is in accord with the Coulomb branch interpretation. As we have seen, though, trying to extract the actual values of all gauge invariant operators in the general case is not possible within the approximation of linearized supergravity. However, this limitation does not invalidate our general prescription — that the values of $\mathcal{O}_{(i_1 \cdots i_k)}$ are equal to the moments of the distribution $\sigma_{D3}(\vec{y})$.

2.4 Spherical shell solution

A multi-center solution of particular interest is obtained by taking the D3-branes to be uniformly distributed over a five dimensional sphere of radius $|\vec{y}| = l$. As is familiar from electrostatics, we expect that inside the sphere the harmonic function will be constant, while outside the sphere it will take the form corresponding to a D3-brane situated at the origin. Indeed by evaluating (2.5) (we drop the 1 as appropriate for comparing with gauge theory) we find:

$$H_{D3}(\vec{y}) = \begin{cases} R^4/l^4 & |\vec{y}| < l\,, \\ R^4/|\vec{y}|^4 & |\vec{y}| > l\,. \end{cases} \tag{2.20}$$

From the metric (2.4), we see that the solution consists of flat ten dimensional Minkowski space inside the shell, and $AdS_5 \times S^5$ outside. Similarly, $C^{(4)}$ has vanishing field strength inside the sphere, and its standard $AdS_5 \times S^5$ value outside. We also note that the divergence in the second derivative of H_{D3} at $|\vec{y}| = l$ can be smoothed out by instead considering a spherical shell of finite thickness. Although the free parameter l appears in the solution, note that its value can be rescaled by a coordinate transformation. Indeed, the proper radial size of the flat space region is R, and so is independent of l.[1]

This supergravity configuration is represented in the gauge theory as a particular point on the Coulomb branch. It is interesting to ask which gauge theory excitations represent the fluctuations of supergravity fields in the flat spacetime region. We have not analyzed this question in detail. However, general features of the AdS/CFT correspondence suggest that bulk physics in the region near the origin is related to the low energy physics of the gauge theory. Thus we expect Minkowski space to control some nontrivial infrared fixed point. This interpretation merits further investigation.

[1]We thank O. Aharony for a discussion on this point.

3. Supergravity backgrounds

When we consider the gauge theory at finite temperature and chemical potential, the vacuum degeneracy is removed. To determine the points in moduli space that are thus singled out, we consider rotating non-extreme backgrounds. In the extreme limit, these solutions are non-rotating but the rotational parameters are retained. These are the nonvanishing moduli, and they allow a simple geometric interpretation. The rotating D3-brane is the main example, given in detail. For future reference, we also note that many extremal solutions have naked singularities on surfaces containing D3- branes. The rotating M2- and M5-branes are qualitatively similar, and the results are stated with less discussion.

3.1 Extreme rotating D3-brane

We are interested in configurations that arise as extreme limits of rotating black 3-branes. The rotation group of the six-dimensional transverse space is the rank three group SO(6), and so there are three independent rotational parameters, denoted l_i, $i = 1, 2, 3$. Before taking the extreme limit these parameters indicate angular momentum in three orthogonal two-planes. The five angular coordinates are chosen as the azimuthal angles $\phi_{1,2,3}$ of these two-planes, and two additional polar angles θ, ψ. The full non-extreme solution is known explicitly and it is reproduced in the appendix A. In the extreme limit $m \to 0$ and $\delta \to \infty$, with $R^4 \sim me^{2\delta}$ fixed, it becomes:

$$
\begin{aligned}
ds_S^2 = {} & H_{D3}^{-\frac{1}{2}} \left[-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right] + H_{D3}^{\frac{1}{2}} f_{D3}^{-1} \frac{dr^2}{\prod_{i=1}^3 (1 + \frac{l_i^2}{r^2})} + \\
& + H_{D3}^{\frac{1}{2}} r^2 \left[\left(1 + \frac{l_1^2 \cos^2\theta}{r^2} + \frac{l_2^2 \sin^2\theta \sin^2\psi}{r^2} + \frac{l_3^2 \sin^2\theta \cos^2\psi}{r^2}\right) d\theta^2 + \cos^2\theta d\psi^2 - \right. \\
& \qquad - 2\frac{l_2^2 - l_3^2}{r^2} \cos\theta \sin\theta \cos\psi \sin\psi d\theta d\psi + \left(1 + \frac{l_1^2}{r^2}\right) \sin^2\theta d\phi_1^2 + \\
& \qquad \left. + \left(1 + \frac{l_2^2}{r^2}\right) \cos^2\theta \sin^2\psi d\phi_2^2 + \left(1 + \frac{l_3^2}{r^2}\right) \cos^2\theta \cos^2\psi d\phi_3^2 \right], \qquad (3.1)
\end{aligned}
$$

$$
C^{(4)} = (H_{D3}^{-1} - 1)\, dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \,, \qquad (3.2)
$$

where:

$$
H_{D3} = 1 + f_D \frac{R^4}{r^4} \,, \qquad (3.3)
$$

$$
f_{D3}^{-1} = \left(\frac{\sin^2\theta}{1 + \frac{l_1^2}{r^2}} + \frac{\cos^2\theta \sin^2\psi}{1 + \frac{l_2^2}{r^2}} + \frac{\cos^2\theta \cos^2\psi}{1 + \frac{l_3^2}{r^2}} \right) \prod_{i=1}^3 \left(1 + \frac{l_i^2}{r^2}\right) . \qquad (3.4)
$$

The main change relative to the non-extreme case is that the metric components of the form $g_{t\phi_i}$ now vanish. This decouples the space within the brane from the space

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transverse to the brane, and it implies that the angular momenta of the solution vanish. Despite these simplifications, the solution remains quite involved. However, the configuration is extreme, and so it must be possible to write it in the general multi-center form given by (2.3)–(2.4). Indeed, upon performing the coordinate change:

$$\begin{aligned}
y_1 &= \sqrt{r^2 + l_1^2}\sin\theta\cos\phi_1 \\
y_2 &= \sqrt{r^2 + l_1^2}\sin\theta\sin\phi_1 \\
y_3 &= \sqrt{r^2 + l_2^2}\cos\theta\sin\psi\cos\phi_2 \\
y_4 &= \sqrt{r^2 + l_2^2}\cos\theta\sin\psi\sin\phi_2 \\
y_5 &= \sqrt{r^2 + l_3^2}\cos\theta\cos\psi\cos\phi_3 \\
y_6 &= \sqrt{r^2 + l_3^2}\cos\theta\cos\psi\sin\phi_3\,.
\end{aligned} \tag{3.5}$$

the solution takes the form (2.3)-(2.4) with the harmonic function H_{D3} given by (3.3). It is straightforward to verify that H_{D3} is indeed a harmonic function. The fact that the coordinates (3.5) greatly simplify the solution was noticed by Russo [14] in the case $l_{2,3} = 0$.

At the present stage the harmonic function (3.3) is still given as a function of the Schwarzchild-like coordinates. Ideally, we would like to rewrite it in terms of the isotropic coordinates $\vec{y}$, and then identify the brane distribution underlying the extreme rotating solution by comparison with the general form (2.5). However, it does not in general appear practical to carry out these steps explicitly. Moreover, it follows from the coordinate change (3.5) that the Schwarzschild-like coordinates only cover a subset of spacetime and so there may even be an obstacle in principle: we do not *a priori* know the completion of the solution.

In view of these difficulties we begin in the following with a simple special case where the underlying physics is simple, and then we proceed towards increasing generality. In each step we first state the result for the distribution function $\sigma_{D3}(\vec{y})$, and then discuss its justification.

One component: Assume $l_2 = l_3 = 0$ but take l_1 arbitrary. In this case the density of branes vanishes outside a disc of radius l_1 in the plane defined by $y_3 = y_4 = y_5 = y_6 = 0$. Moreover, the distribution is uniform on the disc:

$$\sigma_{D3}(\vec{y}) = \frac{1}{\pi l_1^2}\,\Theta\left(l_1 - \sqrt{y_1^2 + y_2^2}\right)\delta^{(4)}(\vec{y}_\perp). \tag{3.6}$$

A first indication of this result follows by inspecting the form of H_{D3} (3.3). In the present special case:

$$r^4 f_{D3}^{-1} = r^2\left(r^2 + l_1^2\cos^2\theta\right)\,, \tag{3.7}$$

and so it is apparent that H_{D3} has at least a quadratic singularity when $r = 0$, for all values of θ. According to (3.5) this translates into the surface $y_3 = y_4 = y_5 = y_6 = 0$ (because $r = 0$) and $y_1^2 + y_2^2 \leq l_1^2$ (because θ is arbitrary). Moreover, a two-dimensional surface charge in six spatial dimensions does indeed give rise to quadratic singularities in the potential.

The result can be justified in detail by using (2.5) to find the value of the harmonic function on any axis perpendicular to the plane of the disc:

$$H_{D3}(\theta = 0) = 1 + R^4 \int_0^{l_1} \frac{1}{(\vec{y}^2 + y'^2)^2} \frac{2y'dy'}{l_1^2} = 1 + \frac{R^4}{\vec{y}^2(\vec{y}^2 + l_1^2)} . \tag{3.8}$$

It is a simple matter to check that this equation agrees with the general form (3.3) when the angular momenta satisfy $l_2 = l_3 = 0$ and the angle $\theta = 0$. It now follows from harmonic analysis that the two expressions agree also when $\theta \neq 0$. More explicitly, the harmonic property and the value of the function at $\theta = 0$ together determines the harmonic function everywhere as:

$$H_{D3} = 1 + \sum_{n=0}^{\infty} \frac{Q_{2n}}{|\vec{y}|^{4+2n}} Y_{2n}(\cos^2\theta) , \tag{3.9}$$

$$Q_{2n} = (-)^n R^4 l_1^{2n} , \tag{3.10}$$

where the $Y_k(\cos^2\theta)$ are the scalar spherical harmonics on S^5, with the normalization convention $Y_k(1) = 1$.

The one component case considered here is the focus of section 4. It is also the special case considered by Russo [14].

Two components: Next, consider the case where $l_3 = 0$, but l_1 and l_2 are arbitrary. The branes are located at $y_5 = y_6 = 0$, and on a three dimensional ellipsoidal surface:

$$\sigma_{D3}(\vec{y}) = \frac{1}{\pi^2 l_1^2 l_2^2} \delta\left(\frac{y_1^2 + y_2^2}{l_1^2} + \frac{y_3^2 + y_4^2}{l_2^2} - 1 \right) \delta^{(2)}(\vec{y}_\perp) . \tag{3.11}$$

This result can be motivated as before, by considering the singularities of the harmonic function. The full justification of the result also proceeds as before. The charged three dimensional surface gives the potential:

$$\begin{aligned} H_{D3}(\theta = \psi = 0) &= 1 + R^4 \int_0^{\frac{\pi}{2}} \frac{2\sin\theta' \cos\theta' d\theta'}{(l_1^2 \sin^2\theta' + l_2^2 \cos^2\theta' + \vec{y}^2)^2} \\ &= 1 + \frac{R^4}{(\vec{y}^2 + l_1^2)(\vec{y}^2 + l_2^2)} , \end{aligned} \tag{3.12}$$

on any axis in the plane spanned by y_5 and y_6. This expression agrees with the general form (3.3) when $l_3 = 0$ and $\theta = \psi = 0$; and then harmonic analysis guarantees that the potential is correct throughout, as before.

The three dimensional surface defined by (3.11) is a closed surface when embedded in the four dimensional space with $y_5 = y_6 = 0$. In fact, it is immediately apparent that it is a generic ellipsoid in four spatial dimensions. However, it is important to note that, in the full six dimensional space, this three dimensional surface does *not* divide space into two regions. In this sense it is a higher dimensional analogue of a *ring*, because a ring divides the two dimensional plane into disconnected regions, leaving the three dimensional space connected (but not simply connected).

In the limit $l_2 \to 0$ we should recover the one component case considered above. According to (3.11) this limit forces $y_{3,4} \to 0$, but $(y_3^2 + y_4^2)/l_2^2$ may remain finite. To see the precise agreement it is easiest to integrate (3.6) and (3.11) with respect to y_3, y_4 and observe that the results match.

Three components (symmetric case): An interesting special case with three nonvanishing components is the symmetric assignment $l_1 = l_2 = l_3 \equiv l$. Here the harmonic function is simply:

$$H_{D3} = 1 + \frac{R^4}{(r^2 + l^2)^2} = 1 + \frac{R^4}{|\vec{y}|^4}\,. \tag{3.13}$$

Thus the harmonic function is *independent* of the rotational parameter, when written in terms of the isotropic coordinate $\vec{y}$.

However, this simple result does not determine the brane distribution unambiguously: the original Schwarzschild-like coordinates only cover the part of spacetime with $r > 0$; and so the equation above can be applied only when $|\vec{y}| > l$. Thus, the underlying brane distribution can be any spherically symmetric distribution with the correct total charge. In particular, the potential in (3.13) could arise from a point source, or alternatively from a *sphere* of radius l. The latter interpretation corresponds to the solution discussed in section 2.4.

Three components (general case): when all three components of the angular momentum are nonvanishing the harmonic function does not exhibit any singularity in the limit of $r \to 0$. This behavior is compatible with a brane distribution that is some five dimensional surface, perhaps with nonvanishing density in its interior. The ellipsoid with the defining equation:

$$\frac{y_1^2 + y_2^2}{l_1^2} + \frac{y_3^2 + y_4^2}{l_2^2} + \frac{y_5^2 + y_6^2}{l_3^2} = 1\,, \tag{3.14}$$

is the surface $r = 0$. It realizes many symmetries of the problem and we suspect that it plays some preferred role. However, as noted above in the special case $l_1 = l_2 = l_3$, the underlying brane distribution is an ambiguous entity. The reason is that the Schwarzschild-like coordinates only apply outside the surface (3.14), and we cannot extend the solution inside this surface without *assuming* a particular distribution of the charges.

3.2 Extreme rotating M5-brane

A distribution of M5-branes is described by the solution:

$$ds^2 = H_{M5}^{-\frac{1}{3}}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2 + dx_5^2\right) + H_{M5}^{\frac{2}{3}} \sum_{i=1}^{5} dy_i^2\,, \quad (3.15)$$

$$C^{\star(3)} = (H_{M5}^{-1} - 1) dt \wedge dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 \wedge dx_5\,. \quad (3.16)$$

In general, the harmonic function H_{M5} is given in terms of the normalized distribution of M5-branes (σ_{M5}) as:

$$H_{M5}(\vec{y}) = 1 + R^3 \int d^5y' \frac{\sigma_{M5}(\vec{y'})}{|\vec{y} - \vec{y'}|^3}\,, \quad (3.17)$$

where the coefficient[2] of the harmonic function is $R^3 = \frac{1}{2} l_p^3 N_{M5}$.

The space transverse to the M5 is five dimensional. Its angular space is therefore classified by the rank two group SO(5), giving two independent angular momenta, parametrized by l_1 and l_2. The azimuthal angles in the planes of these angular momenta are $\phi_{1,2}$, and the remaining angular coordinates are the polar angles θ and ψ. The extreme rotating M5 is given in [21] using Schwarzchild-like coordinates. In the isotropic coordinate system defined by:

$$\begin{aligned}
y_1 &= \sqrt{r^2 + l_1^2} \sin\theta \cos\phi_1 \\
y_2 &= \sqrt{r^2 + l_1^2} \sin\theta \sin\phi_1 \\
y_3 &= \sqrt{r^2 + l_2^2} \cos\theta \sin\psi \cos\phi_2 \\
y_4 &= \sqrt{r^2 + l_2^2} \cos\theta \sin\psi \sin\phi_2 \\
y_5 &= r \cos\theta \cos\psi\,,
\end{aligned} \quad (3.18)$$

it is identical to the canonical solution (3.15)–(3.16) with the harmonic function:

$$H_{M5} = 1 + f_{M5} \frac{R^3}{r^3}\,, \quad (3.19)$$

where:

$$f_{M5}^{-1} = \left(\frac{\sin^2\theta}{1 + \frac{l_1^2}{r^2}} + \frac{\cos^2\theta \sin^2\psi}{1 + \frac{l_2^2}{r^2}} + \cos^2\theta \cos^2\psi\right)\left(1 + \frac{l_1^2}{r^2}\right)\left(1 + \frac{l_2^2}{r^2}\right)\,. \quad (3.20)$$

It is not manifest that (3.19) is a harmonic function, because it is written in terms of Schwarzchild-like coordinates, but this is nevertheless the case.

Next, we identify the underlying brane distribution, assuming that one component of the angular momentum vanishes, say $l_2 = 0$. The singularities of the harmonic

[2] We use units where the eleven dimensional Planck length is given by $l_p = (2\pi g_s)^{\frac{1}{3}} \sqrt{\alpha'}$. However, the precise numerical factors will play no role.

function shows that the branes are confined to the disc defined by $y_3 = y_4 = y_5 = 0$ and $|\vec{y}_{||}| \equiv \sqrt{y_1^2 + y_2^2} \leq l_1$. A simple computation verifies that the precise distribution is:

$$\sigma_{M5}(\vec{y}) = \frac{1}{2\pi l_1 \sqrt{l_1^2 - |\vec{y}_{||}|^2}} \Theta(l_1 - |\vec{y}_{||}|)\, \delta^{(3)}(\vec{y}_\perp)\,, \tag{3.21}$$

where $\vec{y}_\perp$ denote the three components $y_{3,4,5}$. Note that this distribution is *not* uniform: the density of branes diverges at the boundary of the disc. Roughly, the branes form a ring of radius l_1, but not a sharp one: the density falls off smoothly between the peak at the "ring" and the centre.

3.3 Extreme rotating M2-brane

The solution that describes any collection of M2-branes is:

$$ds^2 = H_{M2}^{-\frac{2}{3}}(-dt^2 + dx_1^2 + dx_2^2) + H_{M2}^{\frac{1}{3}} \sum_{i=1}^{8} dy_i^2\,, \tag{3.22}$$

$$C^{(3)} = (H_{M2}^{-1} - 1)\; dt \wedge dx_1 \wedge dx_2\,, \tag{3.23}$$

where H_{M2} is a harmonic function that we write in the form:

$$H_{M2}(\vec{y}) = 1 + R^6 \int d^8y' \frac{\sigma_{M2}(\vec{y}')}{|\vec{y} - \vec{y}'|^8}\,. \tag{3.24}$$

The constant R^6 is related to the number of M2-branes as $R^6 = 8 l_p^6 N_{M2}$.

The space transverse to the M2-brane is 8-dimensional and so there are four independent angular momenta and seven angular coordinates, denoted θ,$\psi_{1,2}$, $\phi_{1,2,3,4}$. Starting from (3.22)–(3.23), the change of coordinates

$$\begin{aligned}
y_1 &= \sqrt{r^2 + l_1^2}\sin\theta\cos\phi_1 \\
y_2 &= \sqrt{r^2 + l_1^2}\sin\theta\sin\phi_1 \\
y_3 &= \sqrt{r^2 + l_2^2}\cos\theta\sin\psi_1\cos\phi_2 \\
y_4 &= \sqrt{r^2 + l_2^2}\cos\theta\sin\psi_1\sin\phi_2 \\
y_5 &= \sqrt{r^2 + l_3^2}\cos\theta\cos\psi_1\sin\psi_2\cos\phi_3 \\
y_6 &= \sqrt{r^2 + l_3^2}\cos\theta\cos\psi_1\sin\psi_2\sin\phi_3 \\
y_7 &= \sqrt{r^2 + l_4^2}\cos\theta\cos\psi_1\cos\psi_2\cos\phi_4 \\
y_8 &= \sqrt{r^2 + l_4^2}\cos\theta\cos\psi_1\cos\psi_2\sin\phi_4
\end{aligned} \tag{3.25}$$

yields the form of the extreme rotating M2 brane given in [21]. This computation also identifies the relevant harmonic function as:

$$H_{M2} = 1 + f_{M2}\,\frac{R^6}{r^6}\,, \tag{3.26}$$

where

$$f_{M2}^{-1} = G_{M2} \prod_{i=1}^{4} \left(1 + \frac{l_i^2}{r^2}\right) \tag{3.27}$$

$$G_{M2} = \frac{\sin^2\theta}{1+\frac{l_1^2}{r^2}} + \frac{\cos^2\theta \sin^2\psi_1}{1+\frac{l_2^2}{r^2}} + \frac{\cos^2\theta \cos^2\psi_1 \sin^2\psi_2}{1+\frac{l_3^2}{r^2}} + \frac{\cos^2\theta \cos^2\psi_1 \cos^2\psi_2}{1+\frac{l_4^2}{r^2}}\,.$$

It is elementary (but tedious) to show that H_2 is indeed harmonic, i.e. it satisfies the Laplace equation.

We have determined the underlying distribution of branes in the case of a single angular momentum parameter l_1, i.e. $l_2 = l_3 = l_4 = 0$. It is:

$$\sigma_{M2}(\vec{y}') = \frac{4(l_1^2 - |\vec{y}_{||}|^2)}{2\pi l_1^4}\, \Theta(l_1 - |\vec{y}_{||}|)\; \delta^{(6)}(\vec{y}_\perp)\,, \tag{3.28}$$

where the two dimensional vector $\vec{y}_{||}$ is within the plane of rotation. Thus, we find a disc with radius l_1 in the $M2$ case too. The distribution is nonuniform, with no sharp peaks. In particular, it is smooth at the boundary of the disc.

4. Thermodynamics of rotating D3-branes

In this section we study the thermodynamics of spinning D3 branes in some detail. We also attempt to relate the results obtained from supergravity to the expected behavior of the dual Yang-Mills theory. Direct calculations in the strongly coupled Yang-Mills theory are difficult, but we will see that in some respects the qualitative behavior agrees with our expectations. In other respects the supergravity theory reveals non-trivial features about the gauge theory. For example, in one limiting case there is evidence that the excitations in the gauge theory are governed by a string theory with a string tension determined by the scale of the Higgs VEVs.

4.1 The supergravity theory

For the sake of simplicity, we focus throughout this section on the case where the D3 branes rotate in only one plane, i.e. when only one of the three angular momentum parameters is nonzero. The metric describing this brane configuration is the special case $l_2 = l_3 = 0$ of the metric discussed in the appendix A. Various thermodynamic properties can be read off from the metric. The number of branes N is related to the AdS curvature scale R by:

$$R^4 = \left(\frac{2}{\pi^4} G_N\right)^{\frac{1}{2}} N = 2m \cosh\delta\, \sinh\delta\,. \tag{4.1}$$

The mass and angular momentum are:

$$M = \frac{\pi^2}{8}\, \frac{L^3 m}{G_N}\, (4\cosh^2\delta + 1)\,, \qquad J = \frac{\pi^2}{4}\, \frac{L^3}{G_N}\, lm\cosh\delta\,, \tag{4.2}$$

where L is the size of each direction along which the branes extend.

The position of the horizon in the coordinates of (3.1) is given by:

$$r_H^2 = \frac{\sqrt{l^4 + 8m} - l^2}{2}\,. \tag{4.3}$$

The entropy, the temperature and the angular velocity of the horizon are then given by [22, 13]:

$$S = \frac{\pi^3}{2}\frac{L^3}{G_N} m r_H \cosh\delta\,, \tag{4.4}$$

$$T = \frac{r_H}{4\pi m \cosh\delta}\sqrt{l^4 + 8m}\,, \tag{4.5}$$

$$\Omega_H = \frac{l r_H^2}{2m \cosh\delta}\,. \tag{4.6}$$

One can verify that these quantities satisfy the thermodynamic relation:

$$TdS = dE - \Omega_H dJ\,. \tag{4.7}$$

In the discussion below we consider the near-extremal limit defined by taking δ large with N fixed. The thermodynamic properties in this limit follow from (4.1)–(4.6); they are:

$$E = \frac{3\pi^2}{8}\frac{L^3}{G_N} m\,, \tag{4.8}$$

$$J = \frac{\pi^2}{8}\frac{L^3}{G_N}\, l\sqrt{2m}\, R^2\,, \tag{4.9}$$

$$S = \frac{\pi^3}{4}\frac{L^3}{G_N}\, r_H\sqrt{2m}\, R^2\,, \tag{4.10}$$

$$T = \frac{1}{2\pi}\frac{r_H\sqrt{l^4 + 8m}}{\sqrt{2m}\, R^2}\,, \tag{4.11}$$

$$\Omega_H = \frac{l r_H^2}{\sqrt{2m}\, R^2}\,, \tag{4.12}$$

where r_H is still given by (4.3).

For purposes of comparing with the dual Yang-Mills theory we are interested in black hole solutions which asymptote to AdS spacetime. These can be obtained from the black brane solutions asymptotic to flat space in two steps. First, one takes the near extremal limit where m, l are small in units of R. Next, following [1], one takes the near horizon limit of this near-extremal solution. Once the black hole solutions have been obtained in this manner though, the condition on m and l being small can be relaxed. One finds that the resulting metric is a solution of the Einstein equations asymptotic to AdS space, for all values of the parameters m and l. Moreover, the thermodynamic properties of these solutions continue to be given by (4.8)–(4.12) for the entire range of parameters (with the energy and angular momentum being defined with respect to asymptotic AdS space).

4.2 The Yang-Mills theory

We now turn to understanding these thermodynamic properties from the Yang-Mills theory point of view. The angular momentum (4.8) corresponds to charge under an SO(2) subgroup of the SO(6) R-symmetry group of the Yang-Mills theory. The thermodynamic quantities in (4.8) are in general complicated functions of the dimensionless parameter l^4/m. We will not be able to reproduce these functions by a direct calculation in the strongly coupled gauge theory. Instead, we will study two limiting cases, when l^4/m tends to zero and infinity, and show how some qualitative features agree with our expectations. We will then comment on the general case towards the end.

4.2.1 The $l^4/m \ll 1$ limit

When $l = 0$, we have a non-rotating black hole. It is well known that in this case the dependence of the entropy on the temperature is accurately given by a calculation in the free field limit, apart from an overall normalisation [22]. This leads one to try to understand the $l^4/m \ll 1$ case in this limit as well. Usually, the Grand Canonical ensemble with fixed chemical potential, rather than fixed charge, is more convenient for studying problems of this kind. So one could hope that the thermodynamics with a small chemical potential can be understood by working in the non-interacting theory.

However, this is an inconsistent starting point: a massless relativistic Bose gas cannot sustain a non-zero chemical potential; any attempt to turn on a chemical potential gives rise to Bose Einstein condensation. Mathematically, a chemical potential gives rise to negative occupational probabilities for some zero modes of the scalar fields.

Despite this fact one can proceed in the following simple-minded manner. We turn on a chemical potential in the non-interacting theory, and neglect the zero-modes in studying the thermodynamics. An elementary computation gives the entropy:

$$S = \frac{2\pi^2}{3} T^3 L^3 N^2 \left(1 + \frac{3}{4\pi^2}\frac{\Omega_H^2}{T^2}\right). \tag{4.13}$$

Here Ω_H denotes the chemical potential in the Yang Mills theory, which is identified with the angular velocity in the supergravity theory (4.12). In contrast, expanding the supergravity formulas (4.8)–(4.12) one finds:

$$S = \frac{\pi^2}{2} T^3 L^3 N^2 \left(1 + \frac{1}{2\pi^2}\frac{\Omega_H^2}{T^2}\right). \tag{4.14}$$

The comparison between the Ω_H independent terms in (4.13) and (4.14) is well known [22]. Here we see that the free field calculation correctly reproduces the functional dependence, in particular the factor N^2, for the subleading term as well. The disagreement in the numerical coefficients is not surprising because and (4.14) correspond to Yang-Mills theory at weak and strong 't Hooft coupling, respectively.

The naive treatment of the chemical potential described above may be better justified than one suspects at first glance. To see this we first note that the need to include interactions once the chemical potential is non-zero has a simple physical explanation from the point of view of D3 branes. If the D3 branes are non-interacting, any attempt to turn on an angular momentum or an angular velocity is unsustainable — the branes simply fly apart in the absence of any forces between them. However, once interactions are turned on and the D3 branes are excited above extremality, they experience a gravitational attraction which could provide the required centripetal force for rotation.

These considerations suggest that, in the Yang-Mills theory, keeping the one loop interactions could be a useful starting point for the analysis. We have not carried out such an analysis. However, one expects that the resulting Higgs VEVs are small when the angular momentum is small and further that the rotational energy is a small fraction of the total energy of the system. This suggests that the effects of the Bose condensate can in fact be neglected when calculating in this limit and that the one-loop analysis should compare favorably with (4.13).

4.2.2 The $l^4/m \gg 1$ limit

As the angular momentum is increased one expects the effects of the Bose condensate to grow. The D3 branes should be typically displaced from the origin, and a reasonable fraction of their total energy should go into the Bose condensate, giving rise to the rotation, while the rest goes into thermal excitations, accounting for the entropy.

First, we consider the strict limit, where $m \to 0$ with l kept fixed. Here we have a very explicit description of the condensate: it corresponds to the configuration discussed in section 3, where the D3 branes are distributed uniformly on a planar disc. One finds from (4.8)–(4.12) that the energy E and angular momentum vanish in this limit, as expected. The entropy and the angular velocity vanish as well, while the temperature is constant. Since the curvature diverges at the horizon in this limit, we should not attribute too much significance to the behaviour of the entropy, temperature and angular velocity. Even so, it is reassuring to note that the entropy approaches zero signalling that the system settles down into the ground state.

Next, consider a slightly non-extremal configuration, for which $m/l^4 \ll 1$. This should be described by a slightly excited version of the uniformly charged disc. For any given angular momentum and total energy there is some consistent distribution of energy between the rotational and thermal excitations that could be determined, in principle, by a variational calculation. The thermal excitations give rise to an attractive gravitational force that in turn provides the centripetal force to spin the D3 branes.

There is a simple estimate that supports this picture. From (4.8) and (4.9) we find that the total energy is related to the angular momentum as:

$$E = \frac{J^2}{2I}\,, \tag{4.15}$$

where I is the "effective moment of inertia" given by:

$$I = \frac{1}{6}M_0 l^2\,, \tag{4.16}$$

and $M_0 = \frac{1}{4}(\frac{2}{G_N})^{1/2} N L^3$ is the total mass of the N D3 branes. (For future reference we note that (4.15) and (4.16) are valid for all values of l^4/m.) By way of comparison, the moment of inertia of a collection of branes, uniformly distributed on a disk of radius l, is:

$$I = \frac{1}{2}M_0 l^2\,. \tag{4.17}$$

This is of the same form as (4.17), but it is numerically smaller as is to be expected: first, the branes should be somewhat denser near the origin as compared to the edges; second, in (4.16) the energy E is the total energy of the system instead of the rotational energy, which should be some fraction of this. The details of this picture should be given by a variational principle.

We should also mention that the moment of inertia is exactly of the form one would expect from a Bose-Einstein condensate in the gauge theory. Equation (4.16) can be rexpressed in terms of the typical vacuum expectation value for the Higgs fields $v \sim l/\alpha'$ as $I \sim N v^2 L^3/g_s$. In the gauge theory we expect a state which carries charge to correspond to a time dependent vacuum expectation value for the Higgs field of the form $\phi \sim v e^{-i\omega t}$. After accounting for a factor of N due to a color trace this gives rise to an energy $E \sim N v^2 \omega^2/g_s$ — exactly in accord with (4.15)–(4.16).

As for the entropy, in the limit $m \ll l^4$ it is given by:

$$S \sim (g_s N)^{1/2} E/v\,, \tag{4.18}$$

where v, as above, is the scale of the VEVs for the two non-zero Higgs fields in the gauge theory. The linear dependence of the entropy on the energy is suggestive. It indicates that the effective theory for low-energy excitations above the Bose condensate is a string theory with a string tension of order $v^2/(g_s N)$. Understanding this string theory is obviously of interest, but it might be challenging since it clearly involves the strongly coupled nature of the gauge theory. In section 5 we will find that the energy regime governed by the effective string theory corresponds in the bulk to a singularity in the metric.

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4.2.3 Concluding remarks

We end this section with some remarks. As was noted above (4.15)–(4.16) are in fact valid everywhere in parameter space. This suggests that l continues to provide a measure of the size of the brane configuration, even away from the limit $l^4 \gg m$. In particular l decreases as the energy increases for a fixed value of the angular momentum, so the brane configuration shrinks towards the origin of moduli space. When the angular momentum approaches zero the branes move to the origin and we recover the description valid when $l^4 \ll m$.

Throughout this section we have considered the Yang-Mills theory living on a three torus and we note that in this case the thermodynamic formulas (4.8)–(4.12) do not indicate a phase transition in the bulk of the parameter space governed by m and l.[3] In contrast, for Yang-Mills theory on a three sphere a phase transition does occur, in the microcanonical ensemble, once the radius of the curvature $r_H \sim R$. This is the direct analogue of the phase transition discussed in [24] for the non-rotating case and is associated with the dominant configuration changing from a black hole in AdS space to one which is localised in ten dimensions. Such a phase transition does not occur in the toroidal case [25], in accord with one's expectations from no-hair theorems. This is easy to see for the rotating case as well. The relevant comparison here is between two seven dimensional black holes, one smeared and the other localised in the S^3 transverse to the plane of rotation.

5. Gauge theory interpretation of the spacetime singularity

In some cases the extreme solutions discussed in section 3 are singular on surfaces containing D3 branes. In this section we relate the appearance of this singularity to a change in the behavior of the gauge theory under renormalization group flow. For simplicity we focus on the extreme D3 brane solution with one non-zero angular momentum parameter, i.e. $l_1 \neq 0, l_{2,3} = 0$. The discussion employs the Schwarzchild-like coordinates of (A.1).

The extreme brane geometry has a singularity at radial coordinate $r = 0$. For small r the curvature invariant $R_{\mu\nu}R^{\mu\nu}$ diverges like $l^2/(R^4 r^2)$ [14]. Thus the curvature scale is of order the string scale at radial position $r \sim l/(gN)^{1/2}$.

We will need to relate scales in the bulk and the boundary theories; there are two relations of this kind [26]. The scale of the VEVs for the Higgs fields is related to the size of the brane configuration by $v = l/\alpha'$. A second relation connects the cutoff in the gauge theory to a radial size [27]. In AdS space this takes the form:

$$L = R^2/r\,, \qquad (5.1)$$

[3] The phase transition discussed in [23] occurs as a function of the temperature and chemical potential at a point which lies on the boundary of the parameter space spanned by l, m.

where r and L are the radial position in the bulk and length scale on the boundary respectively. This relation is also the appropriate one when supergravity modes are used as probes [20, 26], or when the string world sheet *ansatz* is used to evaluate Wilson loops in the bulk theory [28, 29]. In the latter case (5.1) relates the size of the loop on the boundary to the minimum radial position to which it meanders inwards in the bulk.

The brane geometry under discusion here is not AdS. However, when $r_{min} \gg l$ one can still use the AdS geometry in estimating the dynamical scale — thus the $\mathcal{N} = 4$ UV theory governs the behavior of the gauge theory. At $r_{min} \sim l$ the full geometry comes into play; this corresponds to a scale $L \sim R^2/l$ in the gauge theory. At this point the qualitative behavior of a probing string world sheet changes: a rough estimate shows that a further decrease in r_{min} does not lead to a substantial increase in the size of the loop. As a result when $r_{min} = l/(gN)^{1/2}$ — the radial position at which the curvature is of string scale — the loop is still of order $L \sim R^2/l$.

Let us pause to note that in terms of energies on the boundary the approximately AdS geometry continues to suffice until an energy scale $E \sim \frac{l}{\sqrt{gN\alpha'}}$. This is lower than the scale of the VEVs for some of the Higgs fields v. Thus we learn, from the supergravity calculation, that in this regime the Wilson loop continues to behave as in the superconformal $\mathcal{N} = 4$ theory; in particular the energy between two static color sources scales like $E \sim 1/L$.

We now turn to our main interest, the spacetime singularity. It is clear that from the above arguments that from the gauge theory point of view the singularity is related to physics at the energy scale $E_{sing} = \frac{l}{\sqrt{gN\alpha'}}$. In section 4.2.2 we studied the low-energy excitations of the gauge theory using thermodynamics and presented evidence that it is governed by an effective string theory. The string tension of this effective theory is set by E_{sing}. An interesting picture therefore emerges. The high energy behavior of the gauge theory is characteristic of the ultraviolet behavior of the superconformal $\mathcal{N} = 4$ theory. Its low-energy behavior is instead governed by an effective string theory. The cross-over between the two should occur at a scale of order the string tension, which agrees with the radial location of the singularity in the bulk.

Clasical supergravity is clearly inadequate for going "past" the singularity. In contrast, the gauge theory is clearly valid at lower energies as well. In fact at the scale $E \sim l/(\sqrt{N}\alpha')$ all the nonabelian gauge bosons become massive and the theory reduces to weakly coupled $\mathrm{U}(1)^{N-1}$ gauge theory.

6. Discussion

We conclude the paper with a few comments on other recent work.

Rotating D3-brane solutions have been analyzed in the computation of glueball masses in [14, 30]. Taking the angular momentum parameters to be large allows one to decouple certain unwanted states from the theory. According to our results this limit amounts to considering a collection of D3-branes distributed on a large disk of radius l. This corresponds to a specific symmetry breaking pattern of the SU(N) gauge theory, still leaving a nontrivial theory. Its precise nature will be relevant for the interpretation of the result given in [14, 30].

Some aspects of the AdS/CFT correspondence on the Coulomb branch were studied in [9]. In particular, one can go to a point in moduli space with unbroken $\mathrm{SU}(N-2)\times\mathrm{U}(1)\times\mathrm{U}(1)$ gauge symmetry and compute the effective action of the U(1) gauge fields. The result can be interpreted as the interaction between two D3-branes in $AdS_5 \times S^5$ due to the exchange of supergravity quanta. Linearized supergravity is adequate here since for large N the presence of the two D3-branes in the bulk is a small perturbation of the background. By contrast, in our examples we consider geometries which are large deformations of $AdS_5 \times S^5$ in the bulk.

Finally we mention [23], on the thermodynamics of rotating D3-branes. This work has some overlap with section 4, which was essentially completed when [23] appeared.

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A. The nonextreme rotating D3-brane

The black hole solutions with one or several rotational parameters are quite complicated in general. In [31, 13] a large class of solutions were found that correspond to rotating fundamental strings in arbitrary dimension D. After duality transformations the solutions with $D = 7$ can be brought into a form where the only excited U(1) field is the one coupling to the charge of D3-branes. Taking into account the various scalar fields that are also present, the general rotating D3-brane solution in 10 dimensions can be computed. The resulting metric is:

$$
\begin{aligned}
ds_S^2 = {} & H_{D3}^{-\frac{1}{2}}\Big[-H_{D3}dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + f_D\frac{2m}{r^4}\Big(\cosh\delta dt - \\
& \qquad -l_1\sin^2\theta d\phi_1 - l_2\cos^2\theta\sin^2\psi d\phi_2 - l_3\cos^2\theta\cos^2\psi d\phi_3\Big)^2\Big] + \\
& + H_{D3}^{\frac{1}{2}} f_{D3}^{-1}\frac{dr^2}{\prod_{i=1}^3\left(1+\frac{l_i^2}{r^2}\right) - \frac{2m}{r^4}} +
\end{aligned}
$$

$$
+H_{D3}^{\frac{1}{2}} r^2 \left[\left(1 + \frac{l_1^2 \cos^2\theta}{r^2} + \frac{l_2^2 \sin^2\theta \sin^2\psi}{r^2} + \frac{l_3^2 \sin^2\theta \cos^2\psi}{r^2} \right) d\theta^2 + \right.
$$
$$
+ (1 + \frac{l_2^2}{r^2} \cos^2(\psi) + \frac{l_3^2}{r^2} \sin^2(\psi) d\psi^2 +
$$
$$
- 2\frac{l_2^2 - l_3^2}{r^2} \cos\theta \sin\theta \cos\psi \sin\psi d\theta d\psi + \left(1 + \frac{l_1^2}{r^2}\right) \sin^2\theta d\phi_1^2 +
$$
$$
\left. + \left(1 + \frac{l_2^2}{r^2}\right) \cos^2\theta \sin^2\psi d\phi_2^2 + \left(1 + \frac{l_3^2}{r^2}\right) \cos^2\theta \cos^2\psi d\phi_3^2 \right] , \quad (A.1)
$$

where:

$$
H_{D3} = 1 + f_D \frac{2m \sinh^2\delta}{r^4} \quad (A.2)
$$
$$
f_{D3}^{-1} = \left(\frac{\sin^2\theta}{1 + \frac{l_1^2}{r^2}} + \frac{\cos^2\theta \sin^2\psi}{1 + \frac{l_2^2}{r^2}} + \frac{\cos^2\theta \cos^2\psi}{1 + \frac{l_3^2}{r^2}} \right) \prod_{i=1}^{3} \left(1 + \frac{l_i^2}{r^2}\right) . \quad (A.3)
$$

The only matter field that is excited is the four-form gauge field:

$$
C^{(4)} = -(H_{D3}^{-1} - 1) \frac{1}{\sinh\delta} dx_1 \wedge dx_2 \wedge dx_3 \wedge
$$
$$
\wedge \Big(\cosh\delta dt - l_1 \sin^2\theta d\phi_1 - l_2 \cos^2\theta \sin^2\psi d\phi_2 - l_3 \cos^2\theta \cos^2\psi d\phi_2 \Big) . \quad (A.4)
$$

In particular, the dilaton field in ten dimensions is constant. The total D3-brane charge is:

$$
R^4 = 2m \sinh\delta \cosh\delta = 4\pi g_s \alpha'^2 N_{D3} . \quad (A.5)
$$

The mass and angular momentum can be read off from the asymptotic geometry. This gives the formulae (4.2).

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